EE 221: Signal Analysis & Systems
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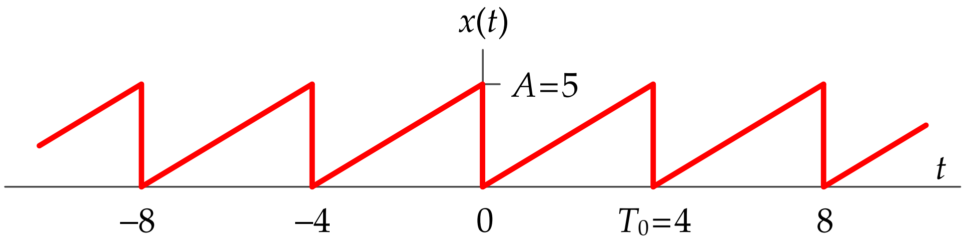
Fourier Series: Practice Problems

Q1. For the following signal $x(t) = \text{rep}_{T_0}\left\{\frac{A}{2}\,\text{saw}\left(\frac{(t-T_0/2)}{T_0}\right)\right\} + \frac{A}{2}$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q1. Answer.

$$\alpha_0 = \frac{A}{2}$$

$$\alpha_n = j\frac{A}{2n\pi}, \quad n \neq 0$$

$$|\alpha_n| = \frac{c_n}{2} = \left|\frac{A}{2n\pi}\right| \qquad \angle\alpha_n = -\theta_n = \begin{cases} \pi/2, & n > 0 \\ 0, & n = 0 \\ -\pi/2, & n < 0 \end{cases}$$

$$a_0 = A$$

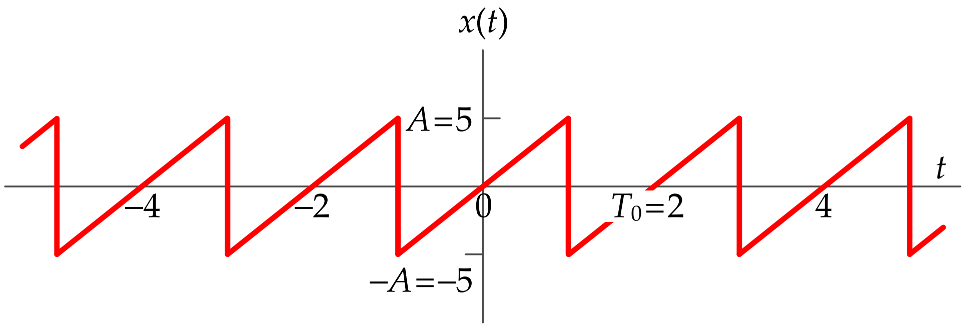
$$a_n = 0, \quad n \neq 0$$

$$b_n = \frac{-A}{n\pi}$$

To see a step-by-step solution of this problem, please watch the YouTube video (https://www.youtube.com/watch?v=_s_P5XsaKdY). Be careful though, and notice the difference between our definition of a_0 and the video's definition of a_0.

To see a step-by-step solution of a similar problem to this one, please watch the YouTube video (https://www.youtube.com/watch?v=yVF_POXZBlw). Be careful though, and notice the difference between our definition of a_0 and the video's definition of a_0.

Q2. For the following signal $x(t) = \text{rep}_{T_0}\left\{A\,\text{saw}\left(\frac{t}{T_0}\right)\right\}$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q2. Answer.

$$\alpha_0 = 0$$

$$\alpha_n = j\frac{A}{n\pi}\cos(n\pi) = j\frac{A}{n\pi}(-1)^n, \quad n \neq 0$$

$$|\alpha_n| = \frac{c_n}{2} = \left|\frac{A}{n\pi}\right| \qquad \angle\alpha_n = -\theta_n = \begin{cases} \pi/2, & n \text{ even} \\ 0, & n = 0 \\ -\pi/2, & n \text{ odd} \end{cases}$$

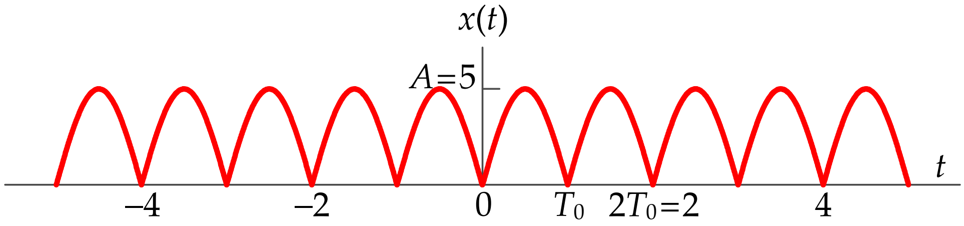
$$a_0 = 0$$

$$a_n = 0, \quad n \neq 0$$

$$b_n = \frac{2A}{n\pi}(-1)^{n+1}$$

To see a step-by-step solution of a similar problem to this one, please watch the YouTube video
(https://www.youtube.com/watch?v=XZF49AZ35bY)
and also the YouTube video
(https://www.youtube.com/watch?v=0NzyqNTRObI).
Be careful though, and notice the difference between our definition of a_0 and the video's definition of a_0, and that the video's C_n corresponds to the complex exponential Fourier series coefficients α_n.

Q3. For the following signal $x(t) = \left|A\sin\left(\frac{\omega_0}{2}t\right)\right|$ (i.e., full-wave rectified signal), determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q3. Solution.
Notice that the fundamental frequency is

$$\omega_0 = \frac{2\pi}{T_0}$$

But the fundamental period T_0 is only half cycle of the original sine wave, rather than its full cycle.

$$\alpha_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jn\omega_0 t}dt = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jn\omega_0 t}dt\,, \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\alpha_n = \frac{1}{T_0}\int_0^{T_0} A\sin\left(\frac{\omega_0}{2}t\right)e^{-jn\omega_0 t}dt\,, \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\alpha_n = \frac{A}{T_0}\int_0^{T_0}\left[\frac{e^{\frac{j\omega_0 t}{2}} - e^{\frac{-j\omega_0 t}{2}}}{2j}\right]e^{-jn\omega_0 t}dt$$

$$\alpha_n = \frac{A}{2jT_0}\int_0^{T_0} e^{\frac{j\omega_0 t}{2}}\,e^{-jn\omega_0 t}dt - \frac{A}{2jT_0}\int_0^{T_0} e^{\frac{-j\omega_0 t}{2}}\,e^{-jn\omega_0 t}dt$$

$$\alpha_n = \frac{A}{2jT_0}\int_0^{T_0} e^{j\left(\frac{1}{2}-n\right)\omega_0 t}dt - \frac{A}{2jT_0}\int_0^{T_0} e^{-j\left(\frac{1}{2}+n\right)\omega_0 t}dt$$

$$\alpha_n = \frac{A}{2jT_0}\left[\frac{1}{j\left(\frac{1}{2}-n\right)\omega_0} e^{j\left(\frac{1}{2}-n\right)\omega_0 t}\right]_0^{T_0} - \frac{A}{2jT_0}\left[\frac{1}{-j\left(\frac{1}{2}+n\right)\omega_0} e^{-j\left(\frac{1}{2}+n\right)\omega_0 t}\right]_0^{T_0}$$

$$\alpha_n = \frac{A}{2jT_0}\left[\frac{e^{j\left(\frac{1}{2}-n\right)\omega_0 T_0} - e^0}{j\left(\frac{1}{2}-n\right)\omega_0}\right] - \frac{A}{2jT_0}\left[\frac{e^{-j\left(\frac{1}{2}+n\right)\omega_0 T_0} - e^0}{-j\left(\frac{1}{2}+n\right)\omega_0}\right]$$

$$\alpha_n = \frac{-A}{2T_0}\left[\frac{e^{j\left(\frac{1}{2}-n\right)2\pi} - 1}{\left(\frac{1}{2}-n\right)\omega_0}\right] - \frac{A}{2T_0}\left[\frac{e^{-j\left(\frac{1}{2}+n\right)2\pi} - 1}{\left(\frac{1}{2}+n\right)\omega_0}\right]$$

But notice that,

$$e^{j\left(\frac{1}{2}-n\right)2\pi} = e^{j(1-2n)\pi} = \cos\big((1-2n)\pi\big) + j\sin\big((1-2n)\pi\big) = -1 + j0, \quad \forall n$$
$$e^{j\left(\frac{1}{2}+n\right)2\pi} = e^{j(1+2n)\pi} = \cos\big((1+2n)\pi\big) + j\sin\big((1+2n)\pi\big) = -1 + j0, \quad \forall n$$

Hence,

$$\alpha_n = \frac{-A}{2T_0}\left[\frac{-2}{\left(\frac{1}{2}-n\right)\omega_0}\right] - \frac{A}{2T_0}\left[\frac{-2}{\left(\frac{1}{2}+n\right)\omega_0}\right]$$

$$\alpha_n = \frac{A}{T_0}\left[\frac{1}{\left(\frac{1}{2}-n\right)\left(\frac{2\pi}{T_0}\right)}\right] + \frac{A}{T_0}\left[\frac{1}{\left(\frac{1}{2}+n\right)\left(\frac{2\pi}{T_0}\right)}\right]$$

$$\alpha_n = A\left[\frac{1}{(1-2n)\pi}\right] + A\left[\frac{1}{(1+2n)\pi}\right]$$

$$\alpha_n = \frac{A}{\pi}\left[\frac{(1+2n)+(1-2n)}{(1-2n)(1+2n)}\right] = \frac{A}{\pi}\left[\frac{2}{1-2n+2n-4n^2}\right]$$

$$\alpha_n = \frac{2A}{\pi(1-4n^2)}, \qquad n = 0, \pm1, \pm2, \ldots$$

$$\alpha_0 = \frac{2A}{\pi}$$

$$\alpha_n = \frac{2A}{\pi(1-4n^2)}, \quad n \neq 0$$

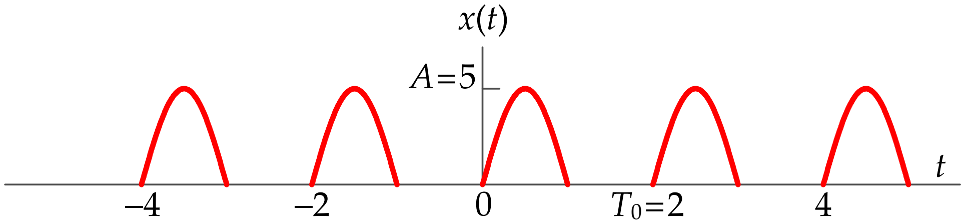
$$|\alpha_n| = \frac{c_n}{2} = \left|\frac{2A}{\pi(1-4n^2)}\right| \qquad \angle\alpha_n = -\theta_n = \pi$$

$$a_0 = \frac{4A}{\pi}$$

$$a_n = \frac{4A}{\pi(1-4n^2)}, \quad n \neq 0$$

$$b_n = 0$$

Q4. For the following signal $x(t) = \sin(\omega_0 t)\,\mathrm{rep}_{T_0}\left\{\mathrm{rect}\left(\frac{(t-T_0/4)}{T_0/2}\right)\right\}$ (i.e., half-wave rectified signal), determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q4. Answer.

$$\alpha_0 = \frac{A}{\pi}$$

$$\alpha_n = \begin{cases} 0 - j\dfrac{A}{4}, & n = 1 \\ \dfrac{A}{\pi(1-n^2)} - j0, & n = 2, 4, 6, 8, \ldots \\ 0 - j0, & n = 3, 5, 7, 9, \ldots \end{cases}$$

$$|\alpha_n| = \frac{c_n}{2} \qquad\qquad \angle\alpha_n = -\theta_n$$

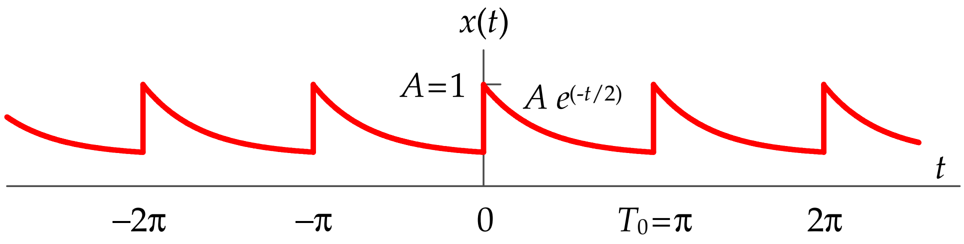
$$a_0 = \frac{2A}{\pi}$$

$$a_n = \begin{cases} \dfrac{2A}{\pi(1-n^2)}, & n = 2, 4, 6, 8, \ldots \\ 0, & n = 1, 3, 5, 7, \ldots \end{cases}$$

$$b_n = \begin{cases} \dfrac{A}{2}, & n = 1 \\ 0, & n = 2, 3, 4, 5, \ldots \end{cases}$$

To see a step-by-step solution of this problem, please watch the YouTube video (https://www.youtube.com/watch?v=SM20fAvKwgY). Be careful though, and notice the difference between our definition of a_0 and the video's definition of a_0.

Q5. For the following signal $x(t) = \text{rep}_{T_0}\left\{A\, e^{(-t/2)}\, \text{rect}\left(\frac{t-T_0/2}{T_0}\right)\right\}$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q5. Solution.

$$\alpha_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jn\omega_0 t}dt = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jn\omega_0 t}dt\,, \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\alpha_n = \frac{1}{T_0}\int_0^{T_0} Ae^{-t/2}e^{-jn\omega_0 t}dt\,, \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\alpha_n = \frac{A}{T_0}\int_0^{T_0} e^{-\left(jn\omega_0+\frac{1}{2}\right)t}dt$$

$$\alpha_n = \frac{A}{T_0}\left[\frac{1}{-\left(jn\omega_0+\frac{1}{2}\right)}e^{-\left(jn\omega_0+\frac{1}{2}\right)t}\right]_0^{T_0}$$

$$\alpha_n = \frac{A}{T_0}\times\frac{1}{-\left(jn\omega_0+\frac{1}{2}\right)}\left[e^{-\left(jn\omega_0+\frac{1}{2}\right)T_0} - e^0\right]$$

$$\alpha_n = \frac{A}{j2\pi n+\frac{\pi}{2}}\left[1 - e^{-\left(j2\pi n+\frac{\pi}{2}\right)}\right]$$

For n integer, the $2\pi n$ added to phase does not change the complex number

$$\alpha_n = \frac{A}{j2\pi n+\frac{\pi}{2}}\left[1 - e^{-\frac{\pi}{2}}\,e^{-j2\pi n}\right] = \frac{A}{j2\pi n+\frac{\pi}{2}}\left[1 - e^{-\frac{\pi}{2}}\,e^{0}\right] = \frac{A\left[1-e^{-\pi/2}\right]}{j2\pi n+\frac{\pi}{2}}$$

Now we perform complex number division with the help of the conjugate

$$\alpha_n = \frac{A\left[1-e^{-\pi/2}\right]}{\frac{\pi}{2}+j2\pi n} \times \frac{\frac{\pi}{2}-j2\pi n}{\frac{\pi}{2}-j2\pi n} = \frac{A\frac{\pi}{2}\left[1-e^{-\pi/2}\right]-jA(2\pi n)\left[1-e^{-\pi/2}\right]}{\left(\frac{\pi}{2}\right)^2+(2\pi n)^2}$$

$$\alpha_n = \frac{A\frac{\pi}{2}\left[1-e^{-\pi/2}\right]-jA(2\pi n)\left[1-e^{-\pi/2}\right]}{\left(\frac{\pi}{2}\right)^2+(2\pi n)^2} = \frac{a_n}{2} - j\frac{b_n}{2}$$

So,

$$a_n = \frac{2A\frac{\pi}{2}\left[1-e^{-\pi/2}\right]}{\left(\frac{\pi}{2}\right)^2+(2\pi n)^2} = \frac{A}{\pi} \times \frac{1}{\left(\frac{1}{2}\right)^2+(2n)^2} \times \left[1-e^{-\pi/2}\right]$$

$$a_n = A \times \frac{2(1-e^{-\pi/2})}{\pi} \times \frac{2}{1+16n^2} = A \times 0.5043 \times \frac{2}{1+16n^2}$$

And,

$$b_n = \frac{2A(2\pi n)\left[1-e^{-\pi/2}\right]}{\left(\frac{\pi}{2}\right)^2+(2\pi n)^2} = \frac{A}{\pi} \times \frac{4n\left[1-e^{-\pi/2}\right]}{\left(\frac{1}{2}\right)^2+(2n)^2}$$

$$b_n = A \times \frac{2(1-e^{-\pi/2})}{\pi} \times \frac{8n}{1+16n^2} = 0.5043\, A\left(\frac{8n}{1+16n^2}\right)$$

Hence,

$$a_0 = 1.0086\, A$$

$$a_n = 0.5043\, A\left(\frac{2}{1+16n^2}\right), \quad n \neq 0$$

$$b_n = 0.5043\, A\left(\frac{8n}{1+16n^2}\right)$$

And for the complex exponential form,

$$\alpha_0 = 0.504\, A$$

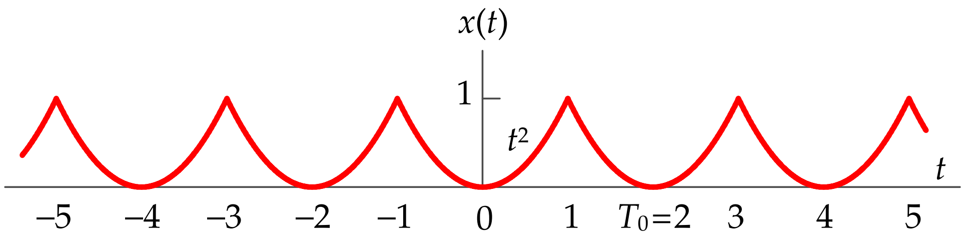
$$\alpha_n = \frac{a_n}{2} - j\frac{b_n}{2} = 0.5043\, A\left(\frac{1}{1+16n^2}\right) - j0.5043\, A\left(\frac{4n}{1+16n^2}\right), \quad n \neq 0$$

$$\alpha_n = \frac{a_n}{2} - j\frac{b_n}{2} = 0.5043\, A\left(\frac{1}{1+16n^2}\right) - j0.5043\, A\left(\frac{4n}{1+16n^2}\right), \quad n \neq 0$$

$$|\alpha_n| = \frac{c_n}{2} = \sqrt{\frac{0.5043^2\, A^2 \times 1}{(1+16n^2)^2} + \frac{0.5043^2\, A^2 \times 16n^2}{(1+16n^2)^2}}$$
$$= 0.5043\, A\left(\frac{1}{\sqrt{1+16n^2}}\right)$$

$$\angle\alpha_n = -\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = \tan^{-1}\left(\frac{-0.5043\, A\left(\frac{8n}{1+16n^2}\right)}{0.5043\, A\left(\frac{2}{1+16n^2}\right)}\right)$$
$$= \tan^{-1}(-4n) = -\tan^{-1}(4n)$$

Q6. For the following signal $x(t) = \text{rep}_2\left\{t^2\,\text{rect}\left(\frac{t}{2}\right)\right\}$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q6. Solution.

$$\alpha_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jn\omega_0 t}dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_0 t}dt\,, n = 0, \pm 1, \pm 2, \ldots$$

$$\alpha_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} t^2 e^{-jn\omega_0 t}dt\,, \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\alpha_n = \frac{1}{T_0}\left[\left(\frac{t^2}{-jn\omega_0} - \frac{2t}{(-jn\omega_0)^2} + \frac{2}{(-jn\omega_0)^3}\right)e^{-jn\omega_0 t}\right]_{-T_0/2}^{T_0/2}$$

$$\alpha_n = \frac{1}{T_0}\left(\left[\left(\frac{T_0^2}{-j4n\omega_0} - \frac{T_0}{(-jn\omega_0)^2} + \frac{2}{(-jn\omega_0)^3}\right)e^{-jn\omega_0 T_0/2}\right] - \left[\left(\frac{T_0^2}{-j4n\omega_0} - \frac{-T_0}{(-jn\omega_0)^2} + \frac{2}{(-jn\omega_0)^3}\right)e^{+jn\omega_0 T_0/2}\right]\right)$$

$$\alpha_n = \frac{1}{2}\left(\left[\left(\frac{4}{-j4n\pi} - \frac{2}{(-jn\pi)^2} + \frac{2}{(-jn\pi)^3}\right)e^{-jn\pi}\right] - \left[\left(\frac{4}{-j4n\pi} - \frac{-2}{(-jn\pi)^2} + \frac{2}{(-jn\pi)^3}\right)e^{+jn\pi}\right]\right)$$

But $e^{-jn\pi}$ is the same as $e^{+jn\pi}$, which is equal to $(-1)^n$, for both n even and odd. Hence,

$$\alpha_n = \frac{1}{2}\times\frac{-4}{(-jn\pi)^2}e^{-jn\pi} = \frac{2}{(n\pi)^2}(-1)^n$$

Q6. Answer.

$$\alpha_0 = 1/3$$

$$\alpha_n = \frac{2 \times (-1)^n}{(n\pi)^2}, \quad n \neq 0$$

$$|\alpha_n| = \frac{c_n}{2} = \left|\frac{2 \times (-1)^n}{(n\pi)^2}\right| = \frac{2}{(n\pi)^2} \qquad \angle\alpha_n = -\theta_n = \begin{cases} 0, & n = 0 \\ 0, & n \text{ even} \\ \pm\pi, & n \text{ odd} \end{cases}$$

$$a_0 = 2/3$$

$$a_n = \frac{4 \times (-1)^n}{(n\pi)^2}, \quad n \neq 0$$

$$b_n = 0$$